JOURNAL OF APPROXIMATION THEORY 79, 169–179 (1994)

On ε-Entropy of Sobolev and Nikolsky Classes in Uniform Metrics on Arbitrary Compacts

A. B. KHODULEV

Keldysh Institute of Applied Mathematics, Moscow, Russia

Communicated by Allan Pinkus

Received February 21, 1992; accepted in revised form March 8, 1993

This paper is devoted to the study of ε-entropy of the Nikolsky classes $H^{\alpha}_{\infty}(I^s)$ in $C(K)$, where K is an arbitrary compact set in I^s. For a connected set K the order of ε-entropy is known to be the same as the order of Komolgorov's ε-dimension. Without connectedness this is not the case. The exact order is given in terms of two functions characterizing "density" and "discontinuity" of the compact K.

1. THEOREMS

Let us fix some notation. In what follows s denotes the dimension of the space where the compact set K lies, $I := [0, 1]$, and $I^s := [0, 1]^s \subset \mathbb{R}^s$. Let K be some nonempty compact subset of I^s. We will consider the ε-entropy of the *bounded* Nikolsky class $CH^{\alpha}_{\infty}(I^s) := H^{\alpha}_{\infty}(I^s) \cap BC(I^s)$ in the seminorm $C(K)$: $\|f(\cdot)\|_{C(K)} := \sup_{x \in K} |f(x)|$. Here BX denotes the unit ball of the Banach space X, and for $0 < \alpha \leqslant 1$, $H^{\alpha}_{\infty}(I^s) := \{f(\cdot) \mid \forall x, y \in I^s \; |f(x) - f(y)| \leqslant d(x, y)^{\alpha}\}$; while for $\alpha > 1$ $H^{\alpha}_{\infty}(I^s) := \{f(\cdot) \in C^{|r|}(I^s) \mid f^{(r)} \in H^{\beta}_{\infty}(I^s),\ \alpha = |r| + \beta,\ r \in \mathbb{Z}^s_+,\ 0 < \beta \leqslant 1\}$. Here $|r| := r_1 + \cdots + r_s$, and $f^{(r)} := \partial^{|r|} f / \partial x_1^{r_1} \cdots \partial x_s^{r_s}$. (For integer α this is the ordinary Sobolev class.) We treat $\mathbb{R}^s$ as a Banach space with the norm l^s_{∞} so that the distance, $d(x, y) := \max_i |x_i - y_i|$. The symbol $U_{\delta}(A)$ for a set $A \subset I^s$ means the δ-neighborhood of the set K, that is, $\{x \in I^s \mid d(x, K) < \delta\}$. We will consider ε-entropy [1], given by

$$\mathscr{H}_{\varepsilon}(C, X) := \log(\min\{\operatorname{Card} M \mid M \subset C,\ \forall c \in C d(c, M, X) \leqslant \varepsilon\}.$$

(Here $d(c, M, X)$ is the distance of a point c from a set M in the Banach space X; $d(c, M, X) := \inf\{d_X(c, y) \mid y \in M\}$. Here and below log means $\log_2$.) To compare it with Kolmogorov's widths we use Kolmogorov's ε-dimension: $\mathscr{K}_{\varepsilon}(C, X) := \min\{k \mid d_k(C, X) \leqslant \varepsilon\}$, where $d_k(C, X)$ is the ordinary Kolmogorov k-width of C in X [2].

0021-9045/94 $6.00

We will write, as usual, $f(\cdot) \ll g(\cdot)$ if $|f(\varepsilon)| < c\,|g(\varepsilon)|$ for some positive constant c and small enough ε; $f(\cdot) \asymp g(\cdot)$ if $f(\cdot) \ll g(\cdot)$ and $g(\cdot) \ll f(\cdot)$; $f(\cdot) \prec g(\cdot)$ if $\lim_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon) = 0$.

To define the mentioned characteristics of "density" and "discontinuity" (functions $n(h)$ and $c(h)$) let us introduce the standard subdivision of the cube I^s and some related notions. For given $h > 0$ and

$$\xi = (n_1, ..., n_s), \qquad n_i \in \mathbb{Z},\ 0 \leqslant n_i < \frac{1}{h},$$

we set

$$B_{\xi,h} := \{(x_1, ..., x_s) \in \mathbb{R}^s \mid n_i h \leqslant x_i < (n_i + 1)h\}.$$

(If $n_i = (1/h) - 1$ replace the last "$<$" with "$\leqslant$" to cover the whole cube.) Ordinarily, we set $h = 2^{-k}$:

$$n(h) := \operatorname{Card}\{B_{\xi,h} \mid B_{\xi,h} \cap K \neq \varnothing\}$$

$$K_h := \bigcup \{B_{\xi,h} \mid B_{\xi,h} \cap K \neq \varnothing\}$$

$$c(h) := \text{the number of connected components of } K_h.$$

Thus, $K_h \supset K$ and K_h is the union of $n(h)$ small cubes with edge of length h.

THEOREM 1. *Using the above notation,*

$$\mathscr{K}_\varepsilon(H^\alpha_\infty(I^s), C(K)) \asymp \mathscr{K}_\varepsilon(CH^\alpha_\infty(I^s), C(K)) \asymp n(\varepsilon^{1/\alpha})$$

This is a simple fact that we will not prove here. Its proof can be easily deduced from our discussion of ε-entropy.

It was shown in [1] that for a connected set K the ε-entropy has the same order. The problem of calculating the ε-entropy in the general case was posed at Tikhomirov's seminar on approximation theory and extremal problems at the Moscow State University.

To estimate ε-entropy the function $n(h)$ alone is insufficient. The result of this paper is the following:

THEOREM 2. *Using the above notation,*

$$\mathscr{K}_\varepsilon(CH^\alpha_\infty(I^s), C(K)) \asymp n(\varepsilon^{1/\alpha}) + \int_{\varepsilon^{1/\alpha}}^{1} \frac{c(h)}{h}\, dh. \tag{1}$$

Proof. Note that for any $a > 0$, $n(h) \leqslant ([a] + 2)^s\, n(ah)$ ($[a]$ is the greatest integer less than or equal to a) and $c(h) \leqslant n(h)$. It follows that for bounded (separated from zero) $a<1$, $n(ah)\asymp n(h)$ and $\int_{ah}^{h}(c(x)/x)\,dx\leqslant\int_{ah}^{h}(n(x)/x)\,dx\ll n(h)$. It means that the order of the right-hand side of (1) is *not changed when* ε *is multiplied by a bounded factor*. Indeed, if $I(\varepsilon)$ means the right-hand side of (1) and $0<a_0<a<1$, then

$$I(a\varepsilon)=n((a\varepsilon)^{1/\alpha})+\int_{(a\varepsilon)^{1/\alpha}}^{\varepsilon^{1/\alpha}}\frac{c(h)}{h}\,dh+\int_{\varepsilon^{1/\alpha}}^{1}\frac{c(h)}{h}\,dh\geqslant\frac{1}{2^s}n(\varepsilon^{1/\alpha})+\int_{\varepsilon^{1/\alpha}}^{1}\frac{c(h)}{h}\,dh$$

and

$$I(a\varepsilon)\ll n(\varepsilon^{1/\alpha})+n(\varepsilon^{1/\alpha})+\int_{\varepsilon^{1/\alpha}}^{1}\frac{c(h)}{h}\,dh\ll I(\varepsilon).$$

Thus, it suffices to prove the theorem for some geometric sequence of ε_n, and, moreover, the ε in the ε-entropy need not be necessarily the same as ε in the right-hand part of (1)—they may differ by a bounded factor (we shall use this possibility later). So we set $\varepsilon_n^{1/\alpha}=2^{-n}$, and denote $h_k:=2^{k-n}$, $k=0,\ldots,n$. Using the above equivalences we obtain:

$$\int_{\varepsilon^{1/\alpha}}^{1}\frac{c(h)}{h}\,dh=\int_{h_0}^{1}\frac{c(h)}{h}\,dh=\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}\frac{c(h)}{h}\,dh\begin{cases}\gg\sum_{k=1}^{n}c(h_k)\\ \ll\sum_{k=1}^{n}c(h_{k-1})\end{cases}.$$

The last two sums differ by $c(h_0)-c(h_n)\ll n(h_0)$. Therefore,

$$\begin{aligned}n(h_0)+\int_{h_0}^{1}\frac{c(h)}{h}\,dh&\asymp n(h_0)+\sum_{k=1}^{n}c(h_k)\\&\asymp n(h_0)+\sum_{k=1}^{n-1}k(c(h_k)-c(h_{k+1}))+(n-1)\,c(h_n)\\&=n(h_0)+\sum_{k=1}^{n-1}k(c(h_k)-c(h_{k+1}))+n-1.\end{aligned}$$

In the last equality we use $c(h_n)=c(1)=1$. (Incidentally, $c(h_{n-1})$ is also equal to 1, but we ignore this fact.) Thus, (1) is equivalent to

$$\mathscr{H}_\varepsilon(CH_\infty^\alpha(I^s),\,C(K)))\asymp n(h_0)+n-1+\sum_{k=1}^{n-1}k(c(h_k)-c(h_{k+1})).\tag{2}$$

This is the statement we shall prove.

Let A be any set containing exactly one point from each cube B_{ξ,h_0} belonging to K_{h_0}; Card $A=n(h_0)$. We number the points of A in "the least distance order." Namely, we assume $A=\{a_i\}_{i=1}^{n(h_0)}$, and for each $i>1$ $d(a_i,A_i)=d(A\backslash A_i,A_i)$, where $A_i=\{a_j\}_{j=1}^{i-1}$. We denote the distance by d_i, $d_i:=d(a_i,A_i)$.

We define the rank $r(a_i)$ of each point a_i as follows. For $i>1$ let $r(a_i)$ be the minimal number r, $r=0, ..., n$, such that the connected component of K_{h_r} containing a_i contains also some previous point a_j, $j<i$. Let, by definition, $r(a_1)=n+1$.

It is clear from the definition of rank that for any point a_i, $i \geqslant 2$, $r:=r(a_i)$:

$$\begin{aligned} h_{r-1} \leqslant d_i \leqslant 2h_r \quad &\text{when} \quad r>0 \\ \text{and} \quad d_i \leqslant 2h_r \quad &\text{when} \quad r=0. \end{aligned} \tag{3}$$

In any case $d_i \ll h_{r(a_i)}$.

We need to count the number of points a_i of a given rank r; let us denote this number by k_r. For any connected component of K_{h_r} consider the first point a_i belonging to this component. Then $r(a_i)>r$. This establishes a one-to-one correspondence between connected components of K_{h_r} (there are $c(h_r)$ ones) and points a_i with $r(a_i)>r$. This means that $c(h_r)=\sum_{q>r} k_q$. Thus

$$\begin{aligned} k_r &= c(h_{r-1}) - c(h_r), \qquad r \geqslant 1, \\ k_0 &= n(h_0) - \sum_{r=1}^{n+1} k_r = n(h_0) - c(h_0). \end{aligned} \tag{4}$$

To find the upper bound for ε-entropy we construct an explicit subdivision of $CH^\alpha_\infty(I^s)$ into sets E_β with diam $E_\beta \ll \varepsilon$. Define E_β as follows:

$$E_\beta := \left\{ f(\cdot) \in CH^\alpha_\infty(I^s) \;\middle|\; \forall a \in A \;\forall j,\; |j|<\alpha : \left[\frac{f^{(j)}(a)}{h_0^{\alpha-|j|}} \right] = \beta(a, j) \right\}.$$

Here $\beta\colon A \times \{j \mid |j|<\alpha\} \to \mathbb{Z}$. For any functions $f_1(\cdot), f_2(\cdot) \in E_\beta$ we have for all $a \in A$, $|f_1^{(j)}(a) - f_2^{(j)}(a)| \leqslant h_0^{\alpha-|j|}$. Then on $K \subset K_{h_0} \subset U_{h_0}(A)$ the Taylor formula yields $|f_1(\cdot) - f_2(\cdot)| \ll h_0^\alpha = \varepsilon$. To estimate the number of nonempty sets E_β we build the set of all possible indexes β (i.e., the set of β for which $E_\beta \neq \varnothing$) step by step considering the vertices a_i in the order established above. Let B_k denote the set of all possible values of $\beta(a_k, \cdot)$ when a_k is fixed and the values of $\beta(a_i, j)$ are fixed for all j and all $i<k$. (The set B_k consists of integer vectors whose components are numbered by the second argument of $\beta(a_k, \cdot)$.) We shall prove that Card $B_k \leqslant M_k$, where M_k does not depend on the values of the previous $\beta(a_i, \cdot)$. Then $\mathrm{Card}\{\beta\} \leqslant \prod M_k$ and $\mathscr{H}_\varepsilon(CH^\alpha_\infty(I^s), C(K)) \ll \log \mathrm{Card}\{\beta\} \leqslant \sum \log M_k$.

So, let us estimate M_k. For $k=1$ the condition $f(\cdot) \in CH^\alpha_\infty(I^s)$ implies that $\log M_1 \asymp \log 1/h_0 = n$. For any other vertex a_k there exists a nearest previous vertex a_i with $d(a_i, a_k) \ll h_r$, where $r=r(a_k)$ (in fact, $d(a_i, a_k) \asymp h_r$). Then the values $f^{(j)}(a_i)$ (that are known to within an error

$\leqslant h_0^{\alpha-|j|}$) allow us to reconstruct the derivatives at a_k by the Taylor formula with an error $\ll h_r^{\alpha-|j|}$. Thus, each component of $\beta(a_k, \cdot)$ has $\ll (h_r/h_0)^{\alpha-|j|}$ possible values, and in total we have

$$\log M_k \ll \begin{cases} 1 & \text{for} \quad r=0 \\ r & \text{for} \quad r>0 \end{cases}.$$

Summing all this quantities and using (4) yields

$$\begin{aligned}\mathscr{H}_\varepsilon(CH_\infty^\alpha(I^s), C(K)) &\ll \sum \log M_k \\ &\ll n + (n(h_0) - c(h_0)) \cdot 1 + \sum_{k=1}^{n} k(c(h_{k-1}) - c(h_k)) \\ &= n + (n(h_0) - c(h_0)) \cdot 1 + \sum_{k=1}^{n} c(h_{k-1}) - c(h_k) \\ &\quad + \sum_{k=1}^{n} (k-1)(c(h_{k-1}) - c(h_k)) \\ &= n + n(h_0) - 1 + \sum_{k=0}^{n-1} k(c(h_k)) - c(h_{k+1})),\end{aligned}$$

which is the right-hand part of (2).

To prove the lower estimate we use a different set A. Here we need $A \subset K$ and the lower bound on distances between points instead of the upper one.

Let $\bar{K}_{h_0,i}$ be an arbitrary connected component of K_{h_0} and $\{B_{\xi_j,h_0}\}$ be the corresponding set of cubes. Let us cut each cube of B_{ξ_j,h_0} in a union of 2^s cubelets with the half-edge: $B_{\xi_j,h_0} = \bigcup_{\gamma \in \{0,1\}^s} B_{2\xi_j+\gamma,h_{-1}}$. Consider the union of all cubelets with the same γ:

$$\bar{K}_{h_0,i,\gamma} := \bigcup_j \{B_{2\xi_j+\gamma,h_{-1}} \subset \bar{K}_{h_0,i} \mid B_{2\xi_j+\gamma,h_{-1}} \cap K \neq \varnothing\}.$$

All cubelets in $\bar{K}_{h_0,i,\gamma}$ are disjoint with distances $\geqslant h_{-1} = h_0/2$ and $\bigcup_\gamma \bar{K}_{h_0,i,\gamma} \cap K = \bar{K}_{h_0,i} \cap K$. Let the number of h_0-cubes in $\bar{K}_{h_0,i}$ be m. Each of them contains at least one cubelet intersecting with K. Therefore by the Dirichlet principle there exists γ such that the set $\bar{K}_{h_0,i,\gamma}$ contains $\geqslant 2^{-s}m$ cubelets. For such γ choose one point $\in K$ from each cubelet $B_{2\xi_j+\gamma,h_{-1}}$. All these points, for all connected components $\bar{K}_{h_0,i}$, constitute the required set A. The number of points in A, n_1, is $\geqslant 2^{-s}n(h_0)$. We order them, $A = \{a_i\}_{i=1}^{n_1}$, and define ranks as above. Then instead of (3), (4) the following properties hold:

$$d_i \geqslant \tfrac{1}{2} h_{r(a_i)} \tag{5}$$

$$k_r = c(h_{r-1}) - c(h_r),\ r \geqslant 1; \qquad k_0 = n_1 - c(h_0). \tag{6}$$

(Here, as above, d_i is the distance between a_i and the nearest previous point; k_r is the number of points of rank r).

To prove the lower bound for the ε-entropy we select a finite-dimensional subset of $CH_\infty^\alpha(I^s)$ and estimate its volume.

LEMMA. *For any $\delta > 0$, $B \subset I^s$ there exists a function $\chi_\delta(B)(\cdot) \in H_\infty^\alpha(I^s)$ such that*

$$\operatorname{supp} \chi_\delta(B) \subset U_\delta(B) \;\&\; \chi_\delta(B)|_B \equiv const \equiv \|\chi_\delta(B)\|_{C(I^s)} = c_1 \delta^\alpha.$$

Proof. I suspect this is well known. However, for completeness a sketch of the proof is given below.

We begin with a nonnegative function $\psi(\cdot) \in C^\infty(\mathbb{R}^s)$ such that (i) $\operatorname{supp} \psi \subset [-1, 1]^s$, (ii) $\sum_{n \in \mathbb{Z}^s} \psi(\cdot - n) \equiv 1$. Such a function in $\mathbb{R}^s$ may be constructed as a product of one-dimensional functions. The required function $\chi_\delta(B)$ can then be built as a sum of properly scaled ψ-functions,

$$\chi_\delta(B)(x) := \sum_{y \in ((\delta/2)\mathbb{Z})^s \cap U_{\delta/2}(B)} c \left(\frac{\delta}{2}\right)^\alpha \psi\left(\frac{x-y}{\delta/2}\right),$$

where c is chosen so that $2^s c\psi(\cdot) \in H_\infty^\alpha(I^s)$. Then any sum of $c\psi(\cdot - n)$ over a subset of $\mathbb{Z}^s$ will be from $H_\infty^\alpha(I^s)$. The desired properties of $\chi_\delta(B)$ immediately follow from the definition. ∎

Without loss of generality we can assume that c_1 is not very large (we need this later).

Let us define a function $\varphi_a(\cdot)$ for each point $a \in A$ as follows: If $a = a_1$ then $\varphi_a :\equiv 1/2$. Otherwise $\varphi_a := c_2^{-1} \chi_{(1/8)h_r}(U_{(1/8)h_r}(A(a)))$, where $r = r(a)$; $A(a)$, when $r > 0$, is the subset $A_{h_{r(a)-1}, i}$ that contains a; for $r(a) = 0$, $A(a) := a$; the constant c_2 will be determined later. The property (5) implies that

$$a \neq b \Rightarrow \operatorname{supp} \varphi'_a \cap \operatorname{supp} \varphi'_b = \varnothing, \tag{7}$$

where $\operatorname{supp} \varphi'$ denotes the union of supports of all derivatives of φ. Indeed, $\operatorname{supp} \varphi'_a \subset U_{(1/4)h_r}(A(a)) \setminus U_{(1/8)h_r}(A(a))$ and all such subsets do not intersect each other. The supports of the functions φ_a can intersect but only for points of different ranks:

$$a \neq b, \qquad \operatorname{supp} \varphi_a \cap \operatorname{supp} \varphi_b \neq \varnothing \Rightarrow r(a) \neq r(b). \tag{8}$$

This also follows from (5). The last consequence of (5):

$$\varphi_{a_i}(a_j) = 0, \qquad \text{for each} \quad j < i. \tag{9}$$

Let us define the set Q of functions

$$Q := \left\{ \sum_{a \in A} w_a \varphi_a(\cdot) \,\middle|\, \forall a\, |w_a| \leqslant 1 \right\}.$$

We shall prove that $Q \subset CH^{\alpha}_{\infty}(I^s)$. The first step is to prove $Q \subset H^{\alpha}_{\infty}(I^s)$. To do this we need some long and not very elegant reasoning. Here it is.

We use the definition of $H^{\alpha}_{\infty}(I^s)$ directly and estimate $|f^{(r)}(x) - f^{(r)}(y)|$ for $f \in Q$, $\alpha = |r| + \beta$, $r \in \mathbb{Z}^s_+$, $0 < \beta \leqslant 1$. Denote $\Delta f := |f^{(r)}(x) - f^{(r)}(y)|$. Then $\Delta f \leqslant \sum_{a \in A} |w_a|\, \Delta\varphi_a \leqslant \sum_{a \in A} \Delta\varphi_a$. We can exclude the term for $a = a_1$ from the last sum because it does not contribute to the sum. For each of the other $\varphi_a = c_2^{-1}\chi_{(1/8)h_r}(U_{(1/8)h_r}(A(a)))$, split the cube I^s into a union of three nonintersecting sets, A_1, A_2, and A_3:

$$A_1(a) := U_{(1/8)h_r}(A(a)); \qquad A_2(a) := U_{(1/4)h_r}(A(a)) \backslash U_{(1/8)h_r}(A(a));$$
$$A_3(a) := I^s \backslash U_{(1/4)h_r}(A(a))$$

Then

$$\varphi_a|_{A_1} \equiv \text{const} = \|\varphi_a\| = c_2^{-1} c_1 (\tfrac{1}{8} h_{r(a)})^{\alpha}$$
$$A_2 \supset \operatorname{supp} \varphi'_a$$
$$\varphi_a|_{A_3} \equiv 0.$$

For all functions φ_a let us see in which set the points x and y lie and split the sum accordingly into nine parts,

$$\sum \Delta\varphi_a = \Sigma_{11} + \Sigma_{12} + \cdots + \Sigma_{33},$$

where $\sum_{jk}$, $j, k = 1, 2, 3$, is the sum of $\Delta\varphi_a$ over all a such that $x \in A_j(a)$, $y \in A_k(a)$. Property (7) means that $\Sigma_{12} + \Sigma_{21} + \Sigma_{22} + \Sigma_{23} + \Sigma_{32}$ (all sums including "2") contains at most two terms (one A_2 including x and maybe another including y), so $\Sigma_{12} + \Sigma_{21} + \Sigma_{22} + \Sigma_{23} + \Sigma_{32} \leqslant 2c_2^{-1} d(x, y)^{\beta}$ because each φ_a is from $c_2^{-1} H^{\alpha}_{\infty}(I^s)$. For $\alpha > 1$ the remaining sums are all zero because $\Delta\varphi_a$ involves only derivatives. So it remains to consider the case $\alpha \leqslant 1$, in which case $\beta = \alpha$.

In any case $\Sigma_{33} = 0$. Also $\Sigma_{11} = 0$ because on A_1 the function φ_a is constant and $\Delta\varphi_a = 0$. To estimate, say Σ_{13} note that for points a in Σ_{13} $\Delta\varphi_a = \|\varphi_a\| = c_1 c_2^{-1} (\tfrac{1}{8} h_{r(a)})^{\beta}$; and, by (8), all points a in that sum have different ranks so it can be estimated by use of the geometric series:

$$\Sigma_{13} < \frac{1}{1 - 2^{\alpha}} \Delta\varphi_{\tilde{a}},$$

where $\tilde{a}$ is the point of maximal rank among all those included in Σ_{13}. But $\varphi\tilde{a} \in c_2^{-1} H_\infty^\alpha(I^s)$, so $\Sigma_{13} < (1/(1-2^{-\alpha}))\, c_2^{-1} d(x,y)^\beta$. Clearly, Σ_{31} has the same bound and finally, for any α,

$$\Delta f \leqslant \sum \Delta\varphi_a < \left(2 + \frac{2}{1-2^{-\alpha}}\right) c_2^{-1} d(x,y)^\beta.$$

So the setting $c_2 = 2 + 2/(1-2^{-\alpha})$ ensures $Q \subset H_\infty^\alpha(I^s)$.

To verify inclusion into $BC(I^s)$ let us estimate $\|f\|$, $f \in Q$. Let $f = \sum w_a \varphi_a(\cdot) \in Q$. Then

$$\forall x \in I^s \qquad |f(x)| = \left|\sum_{a\in A} w_a \varphi_a(x)\right| \leqslant \sum_{\substack{a \in A \\ \varphi_a(x) \neq 0}} |\varphi_a(x)|$$

$$\leqslant \sum_{\substack{a\in A \\ x \in \operatorname{supp}\varphi_a}} \|\varphi_a(\cdot)\| = \sum_{\substack{a \in A \\ x \in \operatorname{supp}\varphi_a \\ a \neq a_1}} c_1 c_2^{-1} (\tfrac{1}{8} h_{r(a)})^\alpha + \tfrac{1}{2}.$$

Property (8) implies that all $r(a)$ in the last sum are different; therefore, we can write the upper estimate for the last sum,

$$\sum_{m=0}^{n} c_1 c_2^{-1} \left(\frac{1}{8} h_m\right)^\alpha \leqslant c_1 c_2^{-1} \frac{8^{-\alpha}}{1-2^{-\alpha}} \leqslant \frac{1}{2},$$

for properly bounded c_1.

Let us now turn to finite-dimensional sets. We have

$$\mathscr{H}_\varepsilon(CH_\infty^\alpha(I^s), C(K)) \geqslant \mathscr{H}_\varepsilon(Q, C(K)) \geqslant \mathscr{H}_\varepsilon(Q, C(A)) = \mathscr{H}_\varepsilon(Q|_A, C(A)),$$

where $Q|_A$ means the set of all functions in Q restricted to the set A. The set A is finite, Card $A = n_1$, so the set $Q|_A$ is in natural one-to-one correspondence with a finite-dimensional set $Q' \subset \mathbb{R}^{n_1}$ $(f(\cdot) \leftrightarrow \langle f(a_i)\rangle)$. Moreover, this correspondence is an isometry between $C(A)$ and $l_\infty^{n_1}$. So the ε-entropy of $Q|_A$ is equal to that of Q' and the last can be estimated by the volume of Q' in $\mathbb{R}^{n_1}$: $\mathscr{H}_\varepsilon(Q', C(A)) \geqslant \log(\text{mes}\, Q'/(2\varepsilon)^{n_1})$. Later we do not distinguish $Q|_A$ and Q'. Let us consider the linear mapping of the cube $\{(w_a) \in \mathbb{R}^{n_1} \mid \forall a \in A\ |w_a| \leqslant 1\}$ onto Q' that maps each vector (w_a) to a function $\sum_{a\in A} w_a \varphi_a(\cdot)$. Property (9) means that the matrix of this mapping $(f_{ij}) = (\varphi_{a_i}(a_j))$ has a triangular form, so its determinant is equal to the product of the diagonal elements, $\varphi_{a_i}(a_i)$, and the volume of the image, mes Q', is

$$\prod_{a\in A} 2\varphi_a(a) = \prod_{a \in A} 2\,\|\varphi_a\| = \left(\prod_{a\in A'} 2c_1 c_2^{-1} (\tfrac{1}{8} h_{r(a)})^\alpha\right) \cdot 1$$

(here $A' = A \setminus \{a_1\}$). The estimate of $\mathscr{H}_\varepsilon$ follows,

$$\mathscr{H}_\varepsilon \geqslant \sum_{a \in A'} (\alpha \log h_{r(a)} - \log \varepsilon + c_3) + \log \frac{1}{2\varepsilon}$$
$$= \sum_{r=0}^{n} k_r \cdot (\alpha \log h_r - \log \varepsilon + c_3) + \log \frac{1}{2\varepsilon}$$

where $k_r = \text{Card}\{a \in A \mid r(a) = r\}$; the values of k_r can be found in (6). We can multiply ε by an appropriate constant (as we have mentioned above, it does not change the order of $\mathscr{H}_\varepsilon$) so that $\alpha \log h_0 - \log \varepsilon + c_3 = 1$ (we will now have $\varepsilon = ch_0^\alpha$). Then for $r > 0$, $\alpha \log h_r - \log \varepsilon + c_3 \asymp r$, $\log 1/2\varepsilon \asymp n$. Thus

$$\mathscr{H}_\varepsilon \geqslant n_1 - c(h_0) + \sum_{k=1}^{n} k(c(h_{k-1}) - c(h_k)) + n$$
$$= n_1 - c(h_0) + \sum_{k=0}^{n-1} (k+1)(c(h_k) - c(h_{k+1})) + n$$
$$= n_1 - c(h_0) + \sum_{k=0}^{n-1} c(h_k) - c(h_{k+1}) + \sum_{k=1}^{n-1} k(c(h_k) - c(h_{k+1})) + n$$
$$= n_1 - 1 + \sum_{k=1}^{n-1} k(c(h_k) - c(h_{k+1})) + n$$
$$\geqslant 2^{-s} n(h_0) - 1 + \sum_{k=1}^{n-1} k(c(h_k) - c(h_{k+1})) + n$$
$$\asymp n(h_0) - 1 + \sum_{k=1}^{n-1} k(c(h_k) - c(h_{k+1})) + n. \quad \blacksquare$$

2. Examples

All examples are one-dimensional.

Example 1. The Cantor Set. Here

$$n(h) \asymp c(h) \asymp h^{-\log_3 2}.$$

Therefore

$$\mathscr{K}_\varepsilon \asymp \mathscr{H}_\varepsilon \asymp \varepsilon^{-\log_3 2/\alpha} = \varepsilon^{-\chi(K)/\alpha},$$

where $\chi(K)$ is the Hausdorff dimension of the Cantor set K.

Moreover, it immediately follows from (1) that if $n(h) \asymp n^{-\gamma}$, $\gamma > 0$, then $\mathcal{K}_\varepsilon \asymp \mathcal{H}_\varepsilon \asymp \varepsilon^{-\gamma/\alpha}$.

EXAMPLE 2.

$$K = \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n = 1, 2, ... \right\}.$$

Here $n(h) \asymp c(h) \asymp h^{-1/2}$, but $\chi(K) = 0$, so

$$\mathcal{K}_\varepsilon \asymp \mathcal{H}_\varepsilon \asymp \varepsilon^{-1/2\alpha} \not\asymp \varepsilon^{-\chi(K)/\alpha}.$$

This illustrates the fact that the considered characteristics relate not to the Hausdorff dimension but to the entropy or metric dimension of the set K [1]:

$$dm(A) := \lim_{\varepsilon \to 0} \frac{\mathcal{H}_\varepsilon(A)}{\log 1/\varepsilon}$$

In this example $dm(K) = 1/2$.

EXAMPLE 3.

$$K = \{0\} \cup \{2^{-n} \mid n = 0, 1, ...\}.$$

Here $n(h) \asymp c(h) \asymp \log h$. We find from (1)

$$\mathcal{K}_\varepsilon \asymp \log \frac{1}{\varepsilon} \qquad \mathcal{H}_\varepsilon \asymp \log^2 \frac{1}{\varepsilon} \succ \mathcal{K}_\varepsilon.$$

In this example the estimate of $\mathcal{K}_\varepsilon$ does not give an estimate of the order of Kolmogorov's width $d_n(H^\alpha_\infty(I^s), C(K))$. This suggests that ε-dimension is more adequate for the problem discussed than n-width.

In spite of the remark after example 1 there do exist compact K with $\chi(K) > 0$ for which $\mathcal{K}_\varepsilon \not\asymp \mathcal{H}_\varepsilon$. One such is constructed in this next example.

EXAMPLE 4. A Cantor-like Set. We begin with the segment

$$\Delta_{01} = I.$$

At the kth step, $k = 1, 2, ...,$ we construct the set of segments

$$\Delta_{ki},\ i = 1, 2, ..., 2^{k(k+1)/2}, \qquad \text{with length} \quad |\Delta_{ki}| = 2^{-k^2}.$$

To do this, let us break each $\Delta_{k-1,i}$ (with length $2^{-(k-1)^2}$) into 2^k segments with equal lengths $2^{-((k-1)^2+k)}$ and select as Δ_k the leftmost part of each subsegment of length 2^{-k^2}. Define the compact K as the intersection of all these segment systems: $K=\bigcap_k \bigcup_i \Delta_{ki}$. As can be easily seen, $\chi(K)=1/2$. For $n(h)$ and $c(h)$ when $h=2^{-m}$ we have

$$n(2^{-m})=\begin{cases} 2^{k(k+1)/2}, & \text{when } k^2-k<m\leqslant k^2, \\ 2^{k(k+1)/2+m-k^2}, & \text{when } k^2<m\leqslant k^2+k \end{cases}$$

$$c(2^{-m})=2^{k(k+1)/2}, \qquad \text{when } k^2-k+1<m\leqslant k^2+k+1.$$

When, for example, $\varepsilon^{1/\alpha}=2^{-k^2}$,

$$\mathscr{K}_\varepsilon \asymp 2^{k(k+1)/2}; \qquad \mathscr{H}_\varepsilon \asymp 2^{k(k+1)/2}\cdot k \asymp \mathscr{K}_\varepsilon \cdot \sqrt{\log \frac{1}{\varepsilon}}.$$

On the other hand, when $\varepsilon^{1/\alpha}=2^{-(k^2-k+1)}$,

$$\mathscr{K}_\varepsilon \asymp 2^{k(k+1)/2}; \qquad \mathscr{H}_\varepsilon \asymp 2^{k(k+1)/2}+2^{k(k-1)/2}\cdot k \asymp \mathscr{K}_\varepsilon.$$

It can be easily deduced from (1) that the case $\mathscr{H}_\varepsilon \succ \mathscr{K}_\varepsilon$ is impossible if $n(h)\gg n^{-\gamma}$, $\gamma>0$. This is the case for all K with nonzero Hausdorff dimension.

REFERENCES

1. A. N. KOLMOGOROV AND V. M. TIKHOMIROV, ε-Entropy and ε-capacity of sets in functional spaces, *Uspekhi Mat. Nauk.* **4**, No. 2(86) (1959), 3–86. [Russian]
2. A. PINKUS, "n-Widths in Approximation Theory," Springer-Verlag, Berlin, 1985.